

Summary.—Experiments are described which provide evidence that a certain fraction of the β -galactosidase molecules of the cell are carried on the ribosomes. This fraction corresponds (in order of magnitude) to one molecule per cell in non-induced, inducible cells and rises to between 10 and 20 per cell for the fully induced and constitutive states. In addition to possessing an apparent higher sedimentation coefficient, the ribosome-bound enzyme molecules are distinguishable from their soluble counterparts in their response to specific anti- β -galactosidase serum. Antiserum precipitates the soluble enzyme without affecting the observed activity. Exposure of the ribosome-bound enzyme to antiserum results in a three- to sixfold rise in activity which is not accompanied by the formation of a precipitable aggregate. It was found that the complex can, however, be precipitated by the addition of an antiserum (chick anti-rabbit) directed against the antibody. The latter reaction suggests a means for the isolation of specific ribosomes.

This work was aided in part by grants to the University of Illinois from the National Institutes of Health, The National Science Foundation, and the Office of Naval Research.

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‡ Aged preparations lose their ability to respond to the specific antibody. One preparation which showed a fourfold increase in the presence of antibody showed no increase after two weeks storage at -20°C .

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³ Bolton, E. T., B. H. Hoyer, and D. B. Ritter in *Microsomal Particles and Protein Synthesis* (New York: Pergamon Press, 1958).

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⁶ Bolton, E. T., R. J. Britten, D. B. Cowie, B. J. McCarthy, K. McQuillen, and R. B. Roberts, *Carnegie Institution of Washington Year Book*, **58** (1959).

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ON THE BOUND STATES OF A GIVEN POTENTIAL*

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Communicated November 29, 1960

It is a fundamental property of any spherically symmetrical potential $V(r)$ for which $\int_0^\infty dr \, r |V(r)|$ exists that there are only a finite number of bound states. This has been expressed by V. Bargmann¹ in the form of the inequality²

$$(2l + 1)n_l < \int_0^\infty dr \, r |V(r)|$$

where n_l is the number of bound states for given l (and magnetic quantum number m_l). The method of derivation of the latter result is sufficiently specialized, however, that it is not easily applied to nonspherically-symmetrical potentials, for

example, or to the tensor forces that appear in the two-nucleon problem. Accordingly, we propose to give another derivation of this inequality, together with some of its extensions.

Any discussion of this subject has as its foundation the elementary fact that a decrease of the potential in some region must lower the energies of the bound states and therefore cannot lessen their number. Thus, we conclude that

$$n_l(V) \leq n_l(-|V|)$$

since the substitution of $-|V(r)|$ for $V(r)$ can either leave the potential unchanged or decrease it. Next, let us replace $-|V(r)|$ by $-\lambda|V(r)|$ with $0 < \lambda \leq 1$. An increase of λ lowers the energies of the bound states and cannot lessen their number. As λ increases from 0, we reach a critical value, λ_1 , at which a bound state first appears at $E = -0$. With further growth of λ , the energy of this state decreases until we reach a second critical value, λ_2 , at which a second bound state appears, and so on. When λ has attained the value unity and

$$\lambda_n \leq 1 < \lambda_{n+1},$$

there are n bound states.

The eigenvalue problem for λ , associated with $E = 0$ and orbital angular momentum l , is given by

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right) u_l(r) = \lambda |V(r)| u_l(r)$$

or, on incorporating the boundary conditions,

$$u_l(r) = \lambda \int_0^\infty dr' g_l(r, r') |V(r')| u_l(r').$$

The Green's function, $g_l(r, r')$, obeys

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right) g_l(r, r') = \delta(r - r')$$

and is explicitly given by

$$g_l(r, r') = \frac{1}{2l+1} r^{<l+1} r^{>-l}.$$

The integral equation can also be written with a real, symmetrical kernel,

$$\lambda^{-1} \phi_l(r) = \int_0^\infty dr' K_l(r, r') \phi_l(r'),$$

where

$$K_l(r, r') = |V(r)|^{1/2} g_l(r, r') |V(r')|^{1/2}$$

and

$$\phi_l(r) = |V(r)|^{1/2} u_l(r).$$

Thus, the eigenvalues of the Hermitian, positive kernel K_l are the reciprocals of the critical numbers $\lambda_1, \lambda_2, \dots$ just described.

The trace of K_l is the sum of the eigenvalues,

$$\sum_1^{\infty} \frac{1}{\lambda_{\alpha}} = \int_0^{\infty} dr K_l(r, r) = \frac{1}{2l+1} \int_0^{\infty} dr r |V(r)|.$$

But, if there are n bound states, $\lambda_1 < \lambda_2 \dots < \lambda_n < 1$, while $0 < \lambda_{\alpha} \leq \infty$, which supplies the inequality

$$\sum_1^{\infty} \frac{1}{\lambda_{\alpha}} \geq \sum_1^n \frac{1}{\lambda_{\alpha}} > n$$

and the theorem

$$n_l(V) \leq n_l(-|V|) < \frac{1}{2l+1} \int_0^{\infty} dr r |V(r)|.$$

When the potential is positive over an appreciable range of r , the upper limit given by this theorem may not be very realistic. Then, one would do better to compare $V(r)$ with a potential that equals $V(r)$ where $V(r) < 0$ but is zero wherever $V(r) > 0$. The same considerations show that

$$n_l(V) < \frac{1}{2l+1} \int dr r |V(r)| \quad V < 0,$$

and now the integration is extended only over the regions of negative $V(r)$. This form of the inequality requires no restriction on the behavior of $V(r)$ in the domain of positive values.

A potential that realizes the upper limit to n_l as closely as one wishes, for any particular l , is given by

$$V(r) = - \sum_{\nu=1}^n V_{\nu} \delta(r - r_{\nu}), \quad V_{\nu} > 0.$$

The eigenvalue problem is equivalent to the n -dimensional determinantal equation

$$\det[\lambda^{-1} \delta_{\mu\nu} - V_{\mu}^{1/2} g_l(r_{\mu}, r_{\nu}) V_{\nu}^{1/2}] = 0.$$

and there are just n eigenvalues. We can now choose each of the ratios $r_{\mu+1}/r_{\mu}$ to be sufficiently large that the nondiagonal elements of the determinant are as small as desired. Furthermore, we make all the diagonal elements equal by requiring of every V_{ν} that

$$V_{\nu} g_l(r_{\nu}, r_{\nu}) = \frac{1}{2l+1} \quad V_{\nu} r_{\nu} \rightarrow 1$$

and in this way attain, with arbitrary precision, a single n -fold degenerate eigenvalue at unity. The resulting situation is one with n bound states of essentially zero binding energy and with $\sum \lambda_{\alpha}^{-1} \cong n$.

The number of bound states is the number of states that lie at or below zero energy, and a similar problem can be posed for any negative energy, $-\kappa^2$. All the previous arguments continue to apply, provided one replaces the Green's function with the one defined by

$$\left(-\frac{d^2}{dr^2} + \kappa^2 + \frac{l(l+1)}{r^2} \right) g_l(r, r', \kappa) = \delta(r - r').$$

The required function is

$$g_i(r, r', \kappa) = \frac{1}{\kappa} S_i(\kappa r_{<}) E_i(\kappa r_{>})$$

where

$$S_i(x) = i^{-l} x j_l(ix)$$

$$E_i(x) = -i^l x h_l(ix)$$

in the notation of spherical Bessel functions. Thus,

$$n_i(E \leq -\kappa^2) < \int_0^\infty dr g_i(rr\kappa) |V(r)| = \int_0^\infty dr r |V(r)| (1/\kappa) S_i(\kappa r) E_i(\kappa r),$$

and
$$\frac{1}{x} S_0(x) E_0(x) = \frac{1}{2x} [1 - e^{-2x}]$$

$$\frac{1}{x} S_1(x) E_1(x) = \frac{1}{2x} \left[1 - \frac{1}{x^2} + e^{-2x} \left(1 + \frac{1}{x} \right)^2 \right]$$

for example. Here again, and in the following, integrations only over the regions of negative V can be used.

The function $g_i(rr\kappa)$, and thereby the integral $\int_0^\infty dr g_i(rr\kappa) |V(r)|$, has two characteristics that are physically necessary as properties of the number of states with any specified l below the energy $-\kappa^2$. It is a monotonically decreasing function of κ and of l . On regarding l together with κ as continuous parameters, we deduce from the Green's function differential equation that for any infinitesimal increase of κ or l ,

$$\delta g_i(rr\kappa) = -\int_0^\infty dr' [2\kappa \delta\kappa + r^{-2}(2l+1)\delta l] (g_i(rr'\kappa))^2 < 0.$$

We shall apply the monotonic dependence on κ to a situation with n bound states for a given l , so that

$$\int_0^\infty dr g_i(rr\kappa_1) |V(r)| > n \geq 1.$$

Then, there is a unique solution of the equation

$$\int_0^\infty dr g_i(rr\kappa_1) |V(r)| = 1,$$

and
$$n_i(E \leq -\kappa_1^2) < 1,$$

which is to say that the lowest bound state, the ground state for the given l , lies above the energy $-\kappa_1^2$, or

$$E_1 > -\kappa_1^2.$$

The deepest of these ground states is the one for $l = 0$, and a lower limit to the ground state energy is obtained by solving the equation

$$\int_0^\infty dr (1 - e^{-2\kappa_1 r}) |V(r)| = 2\kappa_1.$$

For the class of potentials defined by $\int_0^\infty dr |V(r)| < \infty$, we have the crude estimate

$$\kappa_1 < \frac{1}{2} \int_0^\infty dr |V(r)|$$

and
$$E_1 > -\frac{1}{4} \left(\int_0^\infty dr |V(r)| \right)^2.$$

The potential

$$V(r) = -V_1 \delta(r - r_1), \quad V_1 r_1 > 1$$

shows that even this limit can be approached as closely as desired, by choosing $V_1 r_1$ to be sufficiently large. In a similar way, the equation

$$\int_0^\infty dr g_l(r r \kappa_m) |V(r)| = m = 2, 3, \dots \leq n$$

has a unique solution, and

$$n_l(E \leq -\kappa_m^2) < m$$

or

$$E_m > -\kappa_m^2.$$

Accordingly, we have obtained lower limits to the energies of all the bound states of a given potential.³

The transference of these ideas to an arbitrary nonspin-dependent three-dimensional potential $V(r)$ requires only one major modification. The Hermitian, positive kernel of the integral equation

$$\lambda^{-1} \phi(\mathbf{r}) = \int (d\mathbf{r}') K(\mathbf{r} \mathbf{r}') \phi(\mathbf{r}'),$$

is

$$K(\mathbf{r} \mathbf{r}') = |V(\mathbf{r})|^{1/2} G(\mathbf{r} \mathbf{r}') |V(\mathbf{r}')|^{1/2}$$

and the Green's function, which is defined by

$$(-\nabla^2 + \kappa^2) G(\mathbf{r} \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$$

and explicitly presented as

$$G(\mathbf{r} \mathbf{r}') = \frac{e^{-\kappa |\mathbf{r} - \mathbf{r}'|}}{4\pi |\mathbf{r} - \mathbf{r}'|},$$

implies that the kernel is singular on the diagonal. We therefore shift our attention to the iterated kernel, the trace of which is

$$\int (d\mathbf{r})(d\mathbf{r}') (K(\mathbf{r} \mathbf{r}'))^2 = \sum \frac{1}{\lambda_\alpha^2},$$

and deduce for a suitable class of potentials that

$$N(E \leq -\kappa^2) < \frac{1}{(4\pi)^2} \int (d\mathbf{r})(d\mathbf{r}') |V(\mathbf{r})| \frac{e^{-2\kappa |\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|^2} |V(\mathbf{r}')|,$$

which includes an upper limit to the total number of bound states,

$$N < \frac{1}{(4\pi)^2} \int (d\mathbf{r})(d\mathbf{r}') \frac{|V(\mathbf{r})| |V(\mathbf{r}')|}{|\mathbf{r} - \mathbf{r}'|^2}.$$

The latter quantity exists for potentials that decrease more rapidly than $|\mathbf{r}|^{-2}$ as $|\mathbf{r}| \rightarrow \infty$ and that in the neighborhoods of a finite number of points \mathbf{r}_0 are less singular than $|\mathbf{r} - \mathbf{r}_0|^{-2}$. These statements are to be interpreted by such inequalities as

$$|\mathbf{r}| > R, |V(\mathbf{r})| < C|\mathbf{r}|^{-a}, a > 2.$$

A potential of this class that is spherically symmetrical about its only singularity, at the origin, satisfies the condition $\int_0^\infty dr r |V(r)| < \infty$. It should be noted that the upper limit to N , deduced from the individual n_l limits and from the fact that no bound state can occur for $2l + 1 > \int_0^\infty dr r |V|$, is

$$N < \frac{1}{2} \int_0^\infty dr r |V| \left[\int_0^\infty dr r |V| + 1 \right].$$

The energy of the ground state is bounded below by

$$E_1 > -\kappa_1^2$$

where κ_1 is the unique solution of the equation

$$\frac{1}{(4\pi)^2} \int (d\mathbf{x})(d\mathbf{x}') |V(\mathbf{x})| \frac{e^{-2\kappa_1|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|^2} |V(\mathbf{x}')| = 1,$$

given that

$$\frac{1}{(4\pi)^2} \int (d\mathbf{x})(d\mathbf{x}') \frac{|V(\mathbf{x})||V(\mathbf{x}')|}{|\mathbf{x}-\mathbf{x}'|^2} > 1.$$

More generally, when the last integral exceeds the integer N , the solution of the equation

$$\frac{1}{(4\pi)^2} \int (d\mathbf{x})(d\mathbf{x}') V(\mathbf{x}) \frac{e^{-2\kappa_m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|^2} |V(\mathbf{x}')| = m = 1, 2, \dots, N$$

supplies a lower limit to the energy of the m^{th} of the N discrete states,

$$E_m > -\kappa_m^2.$$

There is a simple variant of these procedures that is worth mentioning. A comparison potential to $V(\mathbf{r})$ can be defined that equals $V(\mathbf{r})$ wherever $V(\mathbf{r}) < -\kappa^2$ and equals $-\kappa^2$ in the regions for which $V(\mathbf{r}) \geq -\kappa^2$. The energy values associated with the latter potential are depressed by the amount κ^2 relative to those of the potential that equals $V(\mathbf{r}) + \kappa^2$ wherever this quantity is negative but is zero otherwise. Thus, an upper bound to the total number of states associated with the last potential serves to limit the number of states that lie at or below the energy $-\kappa^2$ for the potential $V(\mathbf{r})$. The explicit statement is

$$N(E \leq -\kappa^2) < \frac{1}{(4\pi)^2} \int (d\mathbf{x})(d\mathbf{x}') \frac{|V(\mathbf{x}) + \kappa^2||V(\mathbf{x}') + \kappa^2|}{|\mathbf{x}-\mathbf{x}'|^2} \Big]_{V+\kappa^2 < 0},$$

and, for the states of given l in a spherically symmetrical potential,

$$n_l(E \leq -\kappa^2) < \frac{1}{2l+1} \int dr r |V(r) + \kappa^2| \Big]_{V+\kappa^2 < 0},$$

from which we obtain lower limits to the energies of the various states. Note that no matter how slowly the potential approaches zero at great distances, there is a finite upper limit to the number of states that lie at or below any energy $-\kappa^2 < 0$. Of course, as $\kappa \rightarrow 0$, this limit will approach infinity if the conditions for a finite total number of states are not satisfied.

As an important example of spin-dependent potentials we consider

$$V(\mathbf{r}) = V_a(r) + V_b(r)S_{12}$$

$$S_{12} = 3 \frac{\boldsymbol{\sigma}_1 \cdot \mathbf{r} \boldsymbol{\sigma}_2 \cdot \mathbf{r}}{r^2} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2,$$

which refers to a pair of particles with spin angular momenta $1/2\sigma$. In the spin singlet state, $S_{12} = 0$ and we need consider only the triplet state, for which $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 = 1$. We then note the algebraic property $(S_{12} + 1)^2 = 9$, so that the triplet eigenvalues of S_{12} are 2 and -4 . Just for simplicity, we shall assume that $V_b(r)$ is everywhere negative, but $V_a(r)$ is not restricted in this way. The notation $A \geq B$ will be used for spin matrices to mean that $A - B$ is a positive matrix, one that can never realize a negative expectation value. Thus,

$$V(\mathbf{r}) \geq V_a(r) + 2V_b(r),$$

and the spin-independent spherically symmetrical potential that equals $V_a + 2V_b$ where this quantity is negative and that equals zero otherwise provides a comparison potential to which the preceding three-dimensional considerations can be applied, since no classification of states is involved.

More detailed results can be obtained by considering specific states, such as the even parity, $J = 1$ states, ${}^3S_1 + {}^3D_1$. The wave function for energy $-\kappa^2$ is described by the pair of radial functions $u_0(r)$, $u_2(r)$ that obey the coupled integral equations

$$u(r) = \int_0^\infty dr' g(rr'\kappa) (-V(r')u(r')).$$

Here,

$$u(r) = \begin{pmatrix} u_0(r) \\ u_2(r) \end{pmatrix}, \quad g = \begin{pmatrix} g_0 & 0 \\ 0 & g_2 \end{pmatrix},$$

and

$$V = \begin{pmatrix} V_a & 2^{1/2}V_b \\ 2^{1/2}V_b & V_a - 2V_b \end{pmatrix}.$$

If we exhibit a matrix, $V_c(r)$, such that

$$V(r) \geq V_c(r)$$

and

$$-V_c(r) \geq 0,$$

the evident matrix generalization of the previous arguments supplies the limit

$$n(E) \leq -\kappa^2 < \int dr \operatorname{tr} [g(rr\kappa) - V_c(r)]$$

where the trace refers to the two-dimensional matrices.

A suitable choice is the comparison potential just described, written as a multiple of the unit matrix, which gives

$$n(E \leq -\kappa^2) < \int_0^\infty dr (g_0(rr\kappa) + g_2(rr\kappa)) [V_a(r) + 2V_b(r)] \Big|_{V_a + 2V_b < 0}$$

and, in particular,

$$n < \frac{6}{5} \int dr r |V_a(r) + 2V_b(r)| \Big|_{V_a + 2V_b < 0}.$$

The latter also implies the alternative bound

$$n(E < -\kappa^2) < \frac{6}{5} \int dr \, r |V_a(r) + 2V_b(r) + \kappa^2| \Big]_{V_a + 2V_b + \kappa^2 < 0}.$$

A somewhat more elaborate treatment follows from the remark that the matrix $V(r)$ defines three regions. In region I, $V_a < -4|V_b|$, and $-V$ is positive definite. Region II is characterized by $2|V_b| > V_a > -4|V_b|$, and here the matrix V is indefinite, while in region III, $V_a > 2|V_b|$ and V is positive definite. For a comparison potential, we use V itself in region I, the multiple $V_a + 2V_b$ of the unit matrix in region II, and zero in region III. There results the upper bound

$$n(E \leq -\kappa^2) < \int_I dr [g_0(rr\kappa)|V_a(r)| + g_2(rr\kappa)|V_a(r) - 2V_b(r)|] + \int_{II} dr (g_0(rr\kappa) + g_2(rr\kappa))|V_a(r) + 2V_b(r)|$$

and

$$n < \int_I dr \, r \left[|V_a(r)| + \frac{1}{5} |V_a(r) - 2V_b(r)| \right] + \frac{6}{5} \int_{II} dr \, r |V_a(r) + 2V_b(r)|.$$

Again, an alternative limit is obtained for $n(E \leq -\kappa^2)$ on replacing $V_a(r)$ with $V_a(r) + \kappa^2$ in the latter formula, with a corresponding redefinition of regions I and II.

In an application to a physical system, such as the deuteron, for which the distribution of energy values is known, these inequalities provide simple bounds on the potential used to represent the data.

* Supported in part by the Air Force Office of Scientific Research (ARDC).

¹ These PROCEEDINGS, 38, 961 (1952).

² The physical constant $2m/\hbar^2$ is absorbed into the definitions of potential and energy.

³ Some remarks in a very recent paper, L. Rosenberg and L. Spruch, *Phys. Rev.*, 120, 474 (1960), footnote 21, indicate that these authors have considered similar questions.

A GLIA-NEURAL THEORY OF BRAIN FUNCTION

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Communicated November 7, 1960

One day, I suppose, someone will find the clue and we shall then realize that we have been watching the missing mechanism at work in every experiment upon the brain that we did, but never recognized it for what it was.

—B. Delisle Burns¹

Theories of brain function abound, ranging from Aristotle's idea that it cools the blood to our present notion that its operations make behavior possible. Theories as to what these operations might be are not scarce either, and they extend from Descartes' idea (in which the pineal gland was supposed to move from left to right to permit humors to flow into one or the other of the brain ventricles) to the present-day almost universal view assigning to neurons alone the critical role. This neuron theory has generated much valuable information about brain function during the